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1993 J. Phys. A: Math. Gen. 26 L75

(http://iopscience.iop.org/0305-4470/26/3/002)

J. Phys. A: Math. Gen. 26 (1993) L75–L82. Printed in the UK

LETTER TO THE EDITOR

The multiparametric non-standard deformation of A_{n-1}

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Received 10 August 1992

Abstract. We introduce a multiparametric generalization of the non-standard R matrix of $sl(n)$. Applying the method of Faddeev–Reshetikhin–Takhtajan (FRT) to this R matrix, we construct the quantum group associated to it. The new feature of this quantum group is that upon a certain choice of the parameters, there can be nilpotent elements in it which is a sign of some hidden superstructure.

By now quantum groups [1–3] have been recognized as a very powerful tool of modern mathematical physics. The q-deformation of the universal enveloping algebras of type A_n, B_n, C_n and D_n, were first given by Drinfeld [1] and Jimbo [2]. Since then there has been a great deal of activity on constructing new examples of quantum groups [4]. For constructing new examples one of the most powerful techniques is the method of Faddeev, Reshetikhin and Takhtajan [3], hereafter called the FRT method in this letter.

A class of new examples, the so-called non-standard or exotic quantum groups were studied in [5]. These quantum groups were based on exotic solutions of quantum Yang–Baxter equation (QYBE). In [5] the quantum groups associated with the non-standard R matrices of the series B_n, D_n and C_n were derived using a method developed in [6]. Obviously the relations derived in [5] are insufficient to characterize an algebra, in particular it is not true to call the relations $(X_i^2 = 0,\ i = 1, \ldots, n)$ 'Serre relations in a particular representation'. Also the relations of [5] are not complete in the sense that the commutation relations of all different simple roots, like X_i and $X_{i\pm 1}$ are not given (and this is what Serre relations do for $\mathrm{Sl}(n)$), which means that an analogue of Poincare–Birkhoff–Wit basis can not be constructed. In this letter we derive the quantum group associated with the multiparametric generalization of the $\mathrm{SL}(n)$ R matrix, using the method of Faddeev–Reshetikhin–Takhtajan (FRT). In particular we show how deformed Serre relations are modified when one considers non-standard solutions of QYBE.

We should emphasize that multiparametric quantization of $\mathrm{GL}(n)$ has been studied by many authors [7–10, 13]. In this letter we show how the quantum group associated with a class of exotic R matrices (i.e. those whose classical limit is not the identity matrix) can be constructed in detail and how it can be multiparametrized consistently. The new feature of this quantum group is that upon a certain choice of the parameters, there can be nilpotent elements in its structure.

0305-4470/93/030075+08$07.50

The structure of this letter is as follows: first we introduce the multiparametric generalization of the $SL(n)$ R matrix, and discuss its properties. Then we apply the method of FRT to the one parametric form of this R matrix and construct the generalization of the universal enveloping algebra of $A_{n-1} \equiv Sl(n)$, which we call $X_q(A_{n-1})$. ($U_q(A_{n-1})$ is then a special case of of $X_q(A_{n-1})$ when we set all q_is equal to q). Next, we show how the structure of $X_q(A_{n-1})$ can be multiparametrized and finally we study the dual algebra, the algebra which is of function algebra type.

Consider the following generalization of the $sl(n)$ R matrix:

$$R(p_{ij}, q_i) = \sum_{i \neq j} p_{ij} e_{jj} \otimes e_{ii} + \sum_{i} q_i e_{ii} \otimes e_{ii} + (q - q^{-1}) \sum_{i<j} e_{ji} \otimes e_{ij} \tag{1}$$

where $p_{ij} = p_{ij}^{-1}$, and each q_i can independently be equal to q or $-q^{-1}$.

This R matrix satisfies QYBE (see appendix) and has $(n(n-1)/2)+1$ parameters. The ordinary R matrix of $sl(n)_q$ is obtained from the above R matrix by setting all p_{ij}s equal to 1 and all q_is to q. It also has the following property:

$$R(p_{ij}, q_i)^{-1} = R(p_{ij}^{-1}, q_i^{-1}).$$

The corresponding Braid matrix ($B = PR$, where P is the permutation matrix) satisfies the quadratic equation (or skein relation in the language of Knot theory):

$$B^2 = (q - q^{-1})B + 1. \tag{3}$$

When a q_i is $-q^{-1}$ we call it a *twisted* parameter.

We start with the following ansatz for the L matrices.

$$L^+ = \sum_{i=1}^{n} k_i e_{ii} + \sum_{i=1}^{n} w(\frac{q_i}{q_{i+1}})^{1/4}(k_i k_{i+1})^{1/2} X_i^+ e_{i,i+1} + \sum_{i<j-1} w(\frac{q_i}{q_j})^{1/4}(k_i k_j)^{1/2} E^+{}_{ij} e_{ij} \tag{4}$$

$$L^- = \sum_{i=1}^{n} k_i^{-1} e_{ii} - \sum_{i=1}^{n} w(\frac{q_i}{q_{i+1}})^{1/4}(k_i k_{i+1})^{-1/2} X_i^- e_{i+1,i} - \sum_{i-1>j} w(\frac{q_i}{q_j})^{1/4}(k_i k_j)^{1/2} E_{ij}^- e_{ij} \tag{5}$$

where $(e_{ij})_{kl} = \delta_{il}\delta_{jk}$ and $w = (q - q^{-1})$.

From the basic equations of the FRT:

$$RL_2^{\pm} L_1^{\pm} = L_1^{\pm} L_2^{\pm} R \qquad RL_2^{+} L_1^{-} = L_1^{-} L_2^{+} R \tag{6}$$

we obtain:

$$k_i k_j = k_j k_i \tag{7}$$

$$(q_i - q_{i+1}) X_i^{\pm 2} = 0 \tag{8}$$

$$k_i X_j^{\pm} = X_j^{\pm} k_i \qquad i \neq j, j+1 \tag{9}$$

$$X_i^{\pm} X_j^{\pm} = X_j^{\pm} X_i^{\pm} \qquad |i-j| \geqslant 2 \tag{10}$$

$$k_i X_i^{\pm} = q_i^{\mp} X_i^{\pm} k_i \tag{11}$$

$$k_{i+1} X_i^{\pm} = q_{i+1}^{\pm 1} X_i^{\pm} k_{i+1} \tag{12}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{k_{i+1} k_i^{-1} - k_i k_{i+1}^{-1}}{q - q^{-1}} \tag{13}$$

$$q_{i+1}^{-1/2} X_i X_{i+1} - q_{i+1}^{1/2} X_{i+1} X_i = -E_{i,i+2} \tag{14}$$

$$q_i^{1/2} X_i E_{i,i+2} = q_i^{-1/2} E_{i,i+2} X_i \tag{15}$$

$$q_{i+2}^{-1/2} X_{i+1} E_{i,i+2} = q_{i+2}^{1/2} E_{i,i+2} X_{i+1}. \tag{16}$$

Serre relations,which are the result of eliminating $E_{i,i+2}$ in equations(14)–(16) after a little algebra take the form:

$$q_i X_i^{\pm 2} X_{i\pm 1}^{\pm} - (1 + q_i q_{i+1}) X_i^{\pm} X_{i\pm 1}^{\pm} X_i^{\pm} + q_{i+1} X_{i\pm 1}^{\pm} X_i^{\pm 2} = 0 \tag{17}$$

with relations between all the elements E_{ij} being supressed. The Hopf structure is then:

$$\Delta(k_i) = k_i \otimes ki \tag{18}$$

$$\Delta(X_i^{\pm}) = (k_i k_{i+1}^{-1})^{1/2} \otimes X_i^{\pm} + X_i^{\pm} \otimes (k_{i+1}^{-1} k_i)^{-1/2} \tag{19}$$

$$\epsilon(k_i) = 1 \qquad \epsilon(X_i^{\pm}) = 0 \tag{20}$$

$$S(k_i) = k_i^{-1} \qquad S(X_i^+) = -(q_i q_{i+1})^{1/2} X_i^+. \tag{21}$$

In the standard case (all $q_i = q$), these relations immediately lead to the $U_q(A_{n-1})$ provided that one identifies $k_{i+1} k_i^{-1}$ with q^{H_i}

It is clear that once k_1 is determined all the other k_is can be determined.

Let us take

$$k_1 = q^{\sum \alpha_i H_i}. \tag{22}$$

To determine α_is we note that in the algebra of $\mathrm{sl}(n)_q$ one has $[H_i, X_j^+] = a_{ij} X_j^+$, where a_{ij} is the Cartan matrix of $\mathrm{sl}(n)$. Comparing (22) with the commutation relations of k_1 and X_js; equations (9)–(12) one obtains the following system of equations for α_is ($i = 1, 2, \ldots n-1$):

$$-\alpha_{i-1} + 2\alpha_i - \alpha_{i+1} = 0. \tag{23}$$

where by convention $\alpha_0 = \alpha_n = 0$. Solving this system of equations gives: $\alpha_i = -(n - i/n)$, and hence

$$k_1 = q^{(-1/n) \sum_{i=1}^{i=N-1} (n-i) H_i}. \tag{24}$$

One then obtains:

$$k_i = q^{(H_1+H_2+H_3+....H_{i-1})} k_1. \tag{25}$$

This is the identification of generators in the FTR approach which leads to the deformation of sl(n) in the Chevalley basis:

$$[H_i, H_j] = 0 \tag{26}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{27}$$

$$[X_i^{\pm}, X_j^{\pm}] = 0 \qquad \text{if} \quad a_{ij} = 0 \tag{28}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \tag{29}$$

together with the generalized Serre relations:

$$X_i^{\pm 2} X_{i\pm1}^{\pm} - (q + q^{-1}) X_i^{\pm} X_{i\pm1}^{\pm} X_i^{\pm} + X_{i\pm1}^{\pm} X_i^{\pm 2} = 0. \tag{30}$$

For the case of $X_q(A_{n-1})$ Lets identify k_{i+1}/k_i with $q^{H_i}\Theta_i$ where H_is are the generators of the Cartan subalgebra and Θ_i $(i = 1, ..., N-1)$ are new generators still in the Cartan subgroup, which we call *twist* generators, and their commutation relations are to be determined. From equations (9)–(12), consistency of the above identification requires the following commutation relations for Θ_i:

$$\Theta_i X_i = \frac{q_i q_{i+1}}{q^2} X_i \Theta_i \tag{31}$$

$$\Theta_i X_{i+1} = \frac{q_i}{q_{i+1}} X_{i+1} \Theta_i \tag{32}$$

$$\Theta_i X_{i-1} = \frac{q}{q_i} X_{i-1} \Theta_i \tag{33}$$

$$\Theta_i X_j = X_j \Theta_i \qquad |i - j| \geqslant 2 \tag{34}$$

which can be written compactly as

$$\Theta_i X_j = \omega_{ij} X_j \Theta_i \tag{35}$$

where

$$\omega_{ij} = \frac{q_i^{\delta_{ij} - \delta_{i-1,j}} q_{i+1}^{\delta_{ij} - \delta_{i+1,j}}}{q^{a_{ij}}}. \tag{36}$$

We call ω_{ij} the *twisting* matrix.

Now it is easy to find the operators Θ_i. Setting $\Theta_i = e^{\sum c_{ik} H_k}$ and $\omega_{ij} = e^{t_{ij}}$ one finds that $\sum c_{ik} a_{kj} = t_{ij}$. Therefore

$$\Theta_i = \prod_{j,k} \omega_{ij}^{a_{jk}^{-1} H_k}. \tag{37}$$

The algebra of non-standard $X_q(A_{n-1})$ now becomes:

$$[H_i, H_j] = 0 \tag{38}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{39}$$

$$[X_i^{\pm}, X_j^{\pm}] = 0 \qquad \text{if} \quad a_{ij} = 0 \tag{40}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{H_i}\Theta_i - q^{-H_i}\Theta_i^{-1}}{q - q^{-1}} \tag{41}$$

$$(q_i - q_{i+1})(X_i^{\pm})^2 = 0 \tag{42}$$

together with the generalized Serre' relations (17): (it is very interesting to note that when a parameter q_i is twisted the Serre relations (17) become vacuous. To circumvent this problem one must quantize the algebra in the Cartan Weyl Basis. This has been done in [11]). The Hopf structure is:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{43}$$

$$\Delta(X_i^{\pm}) = q^{-H_i/2}\Theta_i^{-1/2} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{H_i/2}\Theta_i^{1/2} \tag{44}$$

$$\epsilon(H_i) = \epsilon(X_i^+) = 0 \qquad \epsilon(1) = 1 \tag{45}$$

$$S(H_i) = -H_i \tag{46}$$

$$S(X_i^+) = -(q_i q_{i+1})^{1/2} X_i^+ \tag{47}$$

It is clear that:

$$\Delta(\Theta_i) = \Theta_i \otimes \Theta_i \qquad \epsilon(\Theta_i) = 1 \qquad S(\Theta_i) = \Theta_i^{-1}. \tag{48}$$

It can be easily verified that with the above definitions, all the axioms of Hopf algebra are satisfied [1].

Note the following two special cases:

Case (i). If all q_is $=q$, then $\omega_{ij} = 1$ and all Θ_is can be identified with unity, and the relations (38)–(47) will become the usual relations of $\mathrm{SL}(n)_q$.

Case (ii). If all q_is $=-q^{-1}$, from the form of R matrix, one expects that the quantum group $\mathrm{SL}(n)_q$ will not be twisted, but only its parameter of deformation will become $-q^{-1}$. This is indeed true. Since in this case:

$$\omega_{ij} = \frac{(-q^{-1})^{2\delta_{i,j} - \delta_{i-1,j} - \delta_{i+1,j}}}{q^{a_{ij}}} = \frac{(-q^{-1})^{a_{ij}}}{q^{a_{ij}}} = (-q^{-2})^{a_{ij}}. \tag{49}$$

Hence one can identify Θ_i with $(-q^2)^{H_i}$. Inserting this value of Θ_i in relations (38)–(47) one arrives at the expected result mentioned above.

Now, we use the method of multiparametrization of Hopf algebras proposed in [12]. This method is essentially a generalization of the twisting of Hopf algebras proposed by Reshetikhin in [13].

Let Q_i be operators in the Cartan subgroup, with the following commutation relations:

$$Q_i X_j^{\pm} = s_{ij}^{\pm 1} X_j^{\pm} Q_i \tag{50}$$

where s_{ij} are arbitrary complex numbers.

We define a new coproduct $\Delta_0$as follows:

$$\Delta_0(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{51}$$

$$\Delta_0(X_i^+) = q^{-H_i/2}\Theta_i^{-1/2}Q_i^{-1} \otimes X_i^+ + X_i^+ \otimes q^{H_i/2}\Theta_i^{1/2}Q_i \tag{52}$$

$$\Delta_0(X_i^-) = q^{-H_i/2}\Theta_i^{-1/2}Q_i \otimes X_i^- + X_i^- \otimes q^{H_i/2}\Theta_i^{1/2}Q_i^{-1}. \tag{53}$$

Now it is easily verified that all the commutation relations, equations (38)–(42) are compatible with the new coproduct. That is $\Delta_0[a,b] = [\Delta_0(a), \Delta_0(b)]$, provided that we set

$$s_{ij}s_{ji} = 1 \tag{54}$$

for all i and j. In particular $s_{ii} = \pm 1$, from which it follows that

$$Q_i X_i^{\pm} = \pm X_i^{\pm} Q_i. \tag{55}$$

The above constraint immediately reduces the number of parameters to $r((r-1)/2)$ where $r = n - 1$.

Let us determine the operators Q_i. For the present we restrict ourselves to the solution $s_{ii} = 1$ For this particular choice our ansatz (50) exactly reproduces the results of Reshetikhin in [13].

By a simple use of Baker–Hausdorf formula we find:

$$Q_i = e^{\sum t_{ij} a_{jk}^{-1} H_k} \tag{56}$$

where $s_{ij} = e^{t_{ij}}$ and a sum over repeated indices is understood. Note that due to (54) the matrix t_{ij} is antisymmetric. It has been shown in [12] how this coproduct induces a quasitriangular structure [1] on the Hopf algebra.

The dual algebra [3] is generated by the elements of a matrix T subject to the commutation relations:

$$RT_1T_2 = T_2T_1R \tag{57}$$

and equipped with the Hopf structure:

$$\Delta(t_{ij}) = t_{ik} \otimes t_{kj} \qquad \epsilon(t_{ij}) = \delta_{ij} \qquad S(t_{ij}) = t_{ij}^{-1} \tag{58}$$

Solving equation (57) for the R-matrix (1), we obtain the following commutation relations:

$$(q_i - q_j)t_{ij}^2 = 0 \tag{59}$$

$$t_{ij}t_{kj} = q_j p_{ik} t_{kj} t_{ij} \qquad i < k \tag{60}$$

$$t_{ij}t_{ik} = q_i p_{kj} t_{ik} t_{ij} \qquad j < k \tag{61}$$

$$t_{ij}t_{kl} = p_{ik} p_{lj} t_{kl} t_{ij} \qquad i < k \qquad l < j \tag{62}$$

$$p_{jl}t_{ij}t_{kl} - p_{ik}t_{kl}t_{ij} = (q - q^{-1})t_{il}t_{kj} \qquad i < k \qquad j < l. \tag{63}$$

In the special case that only the parameters q_i, $i = 1, 2 \ldots + m$ are twisted let us write the matrix T in the form:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{64}$$

where A and D are $m \times m$ and $(n-m) \times (n-m)$ blocks respectively. Then it is seen from equations (59) that all the elements of B and C are nilpotent, reminiscent of the fermionic blocks of the quantum supermatrices $\mathrm{SL}(m|n-m)$. But not all the commutation relations are the same as those of the supermatrices.

The quantum vector spaces associated with the R matrix are characterized by

$$x_i x_j = q p_{ij} x_j x_i \qquad i<j \qquad x_i^2 = 0 \qquad \text{for } q_i = -q^{-1} \tag{65}$$

for the quantum vector space $A_n(q_i, p_{ij})$ and

$$\xi_i \xi_j = -q^{-1} p_{ij} \xi_j \xi_i \qquad i<j \qquad \xi^2 = 0 \qquad \text{for } q_i = q \tag{66}$$

for the dual quantum vector space $A_n^*(q_i, p_{ij})$.

It is interesting to note that when a parameter q_i is twisted, the corresponding coordinate in the quantum space x_i becomes nilpotent while the corresponding coordinate in the dual quantum space, ξ_i (which according to Wess and Zumino [14] should be interpreted as the differential form $\mathrm{d}x_i$) is no longer nilpotent.

Clearly the meaning of these non-standard quantum groups and their relation with superalgebras remains to be investigated more rigorously in the future. We discuss the relation of $X_q(\mathrm{sl}(n+m))$ and $U_q(\mathrm{sl}(n|m))$ in a future publication.

We would like to thank V K Dobrev for his very valuable comments on an earlier version of this letter. The work of V Karimipour was partially supported by the Research Institute for Studies in Theoretical Physics and Mathematics, Tehran. This research was supported by the Sharif University of Technology.

Appendix

Although it is straightforward to prove that our R matrix solves QYBE, the calculations are lengthy. For the reader's convenience we give some hints. It is easier to verify that the B matrix $B = PR$ satisfies the Braid equation.

$$B_{12} B_{23} B_{12} = B_{23} B_{12} B_{23}.$$

The explicit form of B is

$$B = \sum_{i \neq j} p_{ji} (e_{ij} \otimes e_{ji}) + \sum_i q_i e_{ii} \otimes e_{ii} + (q - q^{-1}) \sum_{i>j} e_{jj} \otimes e_{ii}. \tag{A.1}$$

Define

$$P_{ij} = \begin{cases} p_{ij} & i \neq j \\ q_i & i = j \end{cases}$$

then since $p_{ij} p_{ji} = 1$ we have

$$P_{ij} P_{ji} = ({q_i}^2 - 1)\delta_{ij} + 1.$$

The Braid equation has the following component form:

$$B_{ab,nm}B_{mc,pk}B_{np,ij} = B_{bc,mn}B_{am,ip}B_{pn,jk} \tag{A.2}$$

from (A.1)

$$B_{ab,nm} = P_{ba}\delta_{am}\delta_{bn} + (q-q^{-1})\delta_{bm}\delta_{an}\theta_{ab} \tag{A.3}$$

where

$$\theta_{ab} = \begin{cases} 1 & a < b \\ 0 & a \geqslant b. \end{cases}$$

Inserting (A.3) in (A.2) and using the following identity for the θ_{ij} symbols

$$\theta_{ij}\theta_{ik} - (\theta_{ij}\theta_{jk} + \theta_{ik}\theta_{kj}) = \theta_{ik}\delta_{kj}$$

one arrives at

$$\begin{aligned}\text{LHS of (A.2)} &= (q-q^{-1})^2 q_b\delta_{ai}\delta_{bj}\delta_{ck}\delta_{bc}\theta_{ca} \\ &\quad + (q-q^{-1})q_b\delta_{ai}\delta_{bj}\delta_{ck}\delta_{bc}\theta_{ca} + (q-q^{-1})\delta_{ai}\delta_{bj}\delta_{ck}\delta_{ab}\theta_{ca} \\ \text{RHS of (A.2)} &= (q-q^{-1})^2 q_b\delta_{ai}\delta_{bj}\delta_{ck}\delta_{ab}\theta_{ca} \\ &\quad + (q-q^{-1})q_b^2\delta_{ai}\delta_{bj}\delta_{ck}\delta_{bc}\theta_{ca} + (q-q^{-1})\delta_{ai}\delta_{bj}\delta_{ck}\delta_{ab}\theta_{ca}.\end{aligned}$$

The two sides will be equal if

$$1 + q_b(q-q^{-1}) = {q_b}^2$$

from which one obtains that any of the q_bs can be q or $-q^{-1}$.

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